THE CONTACT PROBLEM OF THE THEORY OF ELASTICITY FOR THE CASE OF A CIRCULAR AREA OF CONTACT

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References [1-6] and others devoted to this problem have either investigated the problem of the indentation of an elastic body (for instance, a stamp) into an elastic foundation in the form of a half-space with a constant modulus of elasticity, or one which varies according to the law $E = E_\nu z^\nu$ (exact solutions), or they have investigated the indentation into an elastic homogeneous layer (approximate solutions).

The present paper evolves a method, which is not a direct generalization of the methods given in the references, for solving approximately the contact problem for the case of a circular area of contact on an elastic foundation of any sort, the problems listed above being simply particular cases.

Tangential interaction over the area of contact is ignored.

1. We shall first consider the axisymmetric case of the problem.

Suppose that an elastic body (for instance, a stamp) with constants E_0 and μ_0 is pressed into an elastic foundation of any sort for which the expression [7]

$$w(r) = \frac{\theta_1}{2\pi} \int_0^\infty G(ht) J_0(rt)\, dt \tag{1.1}$$

holds, where $J_0(x)$ is a Bessel function; $w(r)$ is the settlement of a point on the surface of the foundation at a distance $r = \sqrt{(x^2 + y^2)}$ from the point of application of a unit force; θ_1 and h are certain positive parameters which characterize the geometric and elastic properties of the foundation.

We shall assume that the continuous function $G(v)$, the analytic form of which is determined by the type of foundation, possesses the property

$$G(\infty) = 1 \tag{1.2}$$

In the case of a homogeneous elastic half-space $G(x) \equiv 1$, and in the case of a layer resting on an absolutely smooth and rigid foundation we have

$$G(x) = 2\sinh^2 x\,(\sinh 2x + 2x)^{-1}, \qquad \theta_m = 2E_m^{-1}(1 - \mu_m^2) \qquad (m = 0, 1) \tag{1.3}$$

In the latter case the thickness of the layer serves as the parameter h.

The form of the function $G(x)$ for a half-space with a modulus of elasticity which varies according the the law $= E_0 \exp(\gamma z)$ is given in [8]. In this case the inverse of γ plays the part of the parameter h.

In order to reduce the present problem to an integral equation for the contact stress $p(r)$, we make use of the formula derived in [7], which gives the settlement $w(r, \rho)$ of points on the surface of the foundation under the action of a vertical load concentrated on the circumference of a circle of radius ρ

$$w_m(r, \rho) = \theta_m \rho \int_0^\infty [G(ht)]^m J_0(rt) J_0(\rho t)\, dt \qquad (m = 0, 1) \tag{1.4}$$

Formula (1.4), which corresponds when $m = 0$ to the case of a half-space, is used below to find the elastic displacements in the body which compresses the foundation.

Following Shtaerman (see, for example, [1, p.175], we easily obtain the integral equation (compare [7])

$$\int_0^a [w_0(r, \rho) + w_1(r, \rho)]\, p(\rho)\, d\rho = z_0 - z_0(r) - z_1(r) \qquad (r \leqslant a) \tag{1.5}$$

Here a is the radius of the area of contact, z_0 is the vertical displacement of the center of gravity of the compressing body, $z = z_0(r)$ and $z = -z_1(r)$ are the equations of the surfaces bounding the compressing body and the foundation respectively (if the surface of the foundation is a plane, then $z_0(r) \equiv 0$).

After transfer to the non-dimensional quantities

$$x = r/h, \qquad \xi = \rho/h, \qquad \alpha = a/h$$

Equation (1.5) becomes

$$\int_0^\alpha K(x, \xi)\, \xi \varphi(\xi)\, d\xi = f(x) \qquad (x \leqslant \alpha) \tag{1.6}$$

Here

$$K(x, \xi) = \sum_{m=0}^{1} \varkappa_m k_m(x, \xi), \qquad \varkappa_m = \frac{\theta_m}{(\theta_0 + \theta_1)} \qquad (m = 0, 1) \tag{1.7}$$

$$k_m(x, \xi) = \int_0^\infty [G(s)]^m J_0(xs) J_0(\xi s)\, ds) \qquad (m = 0, 1) \tag{1.8}$$

$$f(x) = \frac{z_0 - z_0(xh) - z_1(xh)}{\theta_0 + \theta_1} \tag{1.9}$$

The contact stress will be expressed by the solution of Equation (1.6) in the form

$$p(r) = h^{-1}\varphi(r/h) \tag{1.10}$$

The method given below for the solution of Equation (1.6) is based on the approximate representation of its kernel. The representation is obtained as follows. Taking into account that* $G(s) \to 1$ as $s \to \infty$, the function $k_1(x, \xi)$ can be expressed to a high degree of accuracy in the form

$$k_1(x, \xi) \approx \int_0^A G(s) J_0(xs) J_0(\xi s)\, ds + \int_A^\infty J_0(xs) J_0(\xi s)\, ds \tag{1.11}$$

if A is chosen sufficiently large. It is not difficult to obtain from (1.11) the expression [9]:

$$k_1(x, \xi) \approx k_0(x, \xi) - \int_0^A [1 - G(s)] J_0(xs) J_0(\xi s)\, ds \tag{1.12}$$

The last term can easily be expanded into a power series if we make use of the expansion 6.452(2) given in [10]. We shall write this expansion here as follows

$$J_n(xs) J_n(\xi s) = (x\xi)^n \sum_{k=0}^{\infty} \frac{(-1)^k_s{}^{2(k+n)}}{4^{k+n} k! (k+n)!} M_k^{(n)}(x, \xi) \tag{1.13}$$

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* For example, the function $G(x)$ given by Formula (1.3) monotonically approaches unity from below as $x \to \infty$. When $t = 3.2$ it already differs from unity by less than 2 per cent.

where

$$M_k^{(n)}(x,\xi) = x^{2k}(n!)^{-1}F(-k,\,-n-k;\,n+1;\,\xi^2/x^2) =$$
$$= \sum_{j=0}^{k}\binom{k}{j}\binom{k+n}{j}\frac{j!}{(j+n)!}x^{2(k-j)}\xi^{2j} \tag{1.14}$$

We set $n = 0$ and substitute the result into the integrand in (1.12); after carrying out the integration by terms, we obtain an expansion of the second term of the right-hand side of (1.12) in a power series. We retain N terms of this series and substitute (1.12) into (1.7). As a result we obtain the required approximate expression

$$K(x,\xi) \approx k_0(x,\xi) - \varkappa_1 \sum_{k=0}^{N}\frac{(-1)^k C_k M_k^{(0)}(x,\xi)}{4^k (k!)^2} = K^*(x,\xi) \tag{1.15}$$

Here*

$$C_k = \int_0^A [1 - G(s)]\, s^{2k}\, ds \qquad (k = 0,\,1,\,2\ldots)$$

For the integral equation (1.6) with the kernel $K^*(x,\xi)$ we can find an exact solution. This solution we shall also denote by the letter ϕ, by virtue of the fact that it is an approximate solution of the same equation with the kernel $K(x,\xi)$.

To derive the function ϕ is equivalent to finding a function χ related to the latter by the expression

$$\chi(t) = \alpha\varphi(\alpha t), \qquad t = x/\alpha \tag{1.16}$$

and satisfying the equation

$$\alpha\int_0^1 K^*(\alpha t,\,\alpha\tau)\,\tau\chi(\tau)\,d\tau = f(\alpha t) \qquad (t \leqslant 1) \tag{1.17}$$

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* For example, in the case of a foundation in the form of an elastic layer, when (1.3) holds and $A = 3.2$ (see previous note), the first nine values of C_k are

(a)

$C_0 = 1.153$	$C_3 = 27.32$	$C_6 = 10710$
$C_1 = 1.381$	$C_4 = 181.4$	$C_7 = 89400$
$C_2 = 5.023$	$C_5 = 1347$	$C_8 = 769000$

The computations were carried out according to the Gaussian quadrature formula.

We shall try to find a solution to this equation in the form

$$\chi(t) = (1-t^2)^{-1/2} \sum_{m=0}^{\infty} Y_m P_m^*(t) \tag{1.18}$$

Here and in our subsequent work, for the sake of conciseness, we shall adopt the notation

$$P_m^*(t) = P_{2m}(\sqrt{1-t^2}) \qquad (P_n(z) - \text{being a Legendre polynomial}) \tag{1.19}$$

2. It will be shown below that the solution of Equation (1.17) can be expressed by the function (1.18) for a particular choice of Y_m. The proof is based on a remarkable property of Legendre polynomials, namely that

$$\alpha \int_0^1 k_0(\alpha t, \alpha\tau)\, \tau (1-\tau^2)^{-1/2} P_m^*(\tau)\, d\tau = \lambda_m P_m^*(t) \tag{2.1}$$

This property, in a different presentation for the function $k_0(x, \zeta)$, was noted and verified for small values of m in [11]. However, the reference does not contain a conclusive proof (for any given value of m), and for this reason a general formula for calculating the value of λ_m does not exist. Here we shall obviate this problem.

By setting $t = \sqrt{(1 - y^2)}$, and making use of (1.8) and (1.19), we can transform Expression (2.1) to the form

$$\int_0^{\infty} J_0(\alpha s\sqrt{1-y^2})\, ds \int_0^1 J_0(\alpha\tau s)\, P^*(\tau) \frac{\tau d\tau}{\sqrt{1-\tau^2}} = \lambda_m P_{2m}(y)$$

If in the inner integral we make the substitution $r = \sqrt{(1 - \eta^2)}$, we obtain

$$\int_0^{\infty} J_0(\alpha s\sqrt{1-y^2})\, ds\ \frac{1}{2} \int_{-1}^{1} J_0(\alpha s\sqrt{1-\eta^2})\, P_{2m}(\eta)\, d\eta = \lambda_m P_{2m}(y) \tag{2.2}$$

Making use of Formula 6.541(3) in [10], we discover that

$$J_0(z\ \sqrt{1-x^2}) = \sqrt{\frac{2\pi}{z}} \sum_{k=0}^{\infty} \frac{\sigma_k (2k-1)!!}{2^k \cdot k!} J_{\sigma_k}(z)\, P_{2k}(x) \qquad \left(\sigma_k = 2k + \frac{1}{2}\right) \tag{2.3}$$

After evaluating the inner integral in (2.2), making use of expansion (2.3) and the orthogonality of the Legendre polynomials, and having then made the substitution $as = x$, we find after repeated use of expansion (2.3), that Expression (2.2) becomes

$$\frac{\pi\,(2m-1)!!}{2^m\cdot m!}\sum_{k=0}^{\infty}\frac{\sigma_k\,(2k-1)!!}{2^k\cdot k!}P_{2k}(y)\int_0^{\infty}J_{\sigma_m}(x)\,J_{\sigma_k}(x)\frac{dx}{x}=\lambda_m P_{2m}(y) \qquad (2.4)$$

Making use of Formula 4.415(2) of [10], and taking into account the analytic properties of the gamma-function, we obtain

$$\int_0^{\infty}J_{\sigma_m}(x)\,J_{\sigma_k}(x)\frac{dx}{x}=\begin{cases}0 & (k\neq m)\\ (4m+1)^{-1} & (k=m)\end{cases}\qquad \left(\sigma_n=2n+\tfrac{1}{2};\ n=m,\,k\right) \qquad (2.5)$$

Finally, by substituting (2.5) into the left-hand side of (2.4), we can establish the validity of Expression (2.1) and also of the following formula:

$$\lambda_m=0.5\pi 4^{-m}(m!)^{-2}[(2m-1)!!]^2 \qquad (m=0,\,1,\,\ldots) \qquad (2.6)$$

3. We shall prove now that the function (1.18) is a solution of Equation (1.17), and we shall indicate a method of determining the coefficients Y_n.

In order to do so we substitute (1.18) into (1.17), taking into account (1.8), and then make use of Expression (2.1). Then, bearing in mind the orthogonality of the Legendre polynomials, we integrate both sides of Equation (1.17) with respect to t over the range (0.1) with a weight function $(1-t^2)^{-1/2}tP_l^*(t)$; as a result (1.17) becomes

$$\lambda_l(4l+1)^{-1}Y_l-\alpha\varkappa_1\sum_{m=0}^{N-l}Y_m\sum_{\max(m,\,l)}^{N}\frac{(-1)^kC_k\alpha^{2k}}{4^k(k!)^2}B_{mk}^{(l)}=f_l \qquad (3.1)$$

$$(l=0,\,1,\,\ldots,\,N)$$

$$Y_l=(4l+1)\,\lambda_l^{-1}f_l \qquad (l>N) \qquad (3.2)$$

where

$$f_l=\int_0^1\frac{f(\alpha t)\,P_l^*(t)}{\sqrt{1-t^2}}\,t\,dt, \qquad B_{mk}^{(l)}=\sum_{j=m}^{k-l}\binom{k}{j}^2 b_{k-j}^{(l)}b_j^{(m)}$$

$$b_n^{(k)}=\int_0^1\frac{P_k^*(\tau)}{\sqrt{1-\tau^2}}\tau^{2n+1}\,d\tau=\begin{cases}0 & (k>n)\\ \dfrac{(-1)^k(2k)!\,(n-k)!}{2^{k-n}[2(n+k)+1]!!}\dbinom{n}{k}^2 & (n\geqslant k)\end{cases} \qquad (3.3)$$

The solution of the system (3.1) can easily be found in general form by virtue of the fact that its coefficients form an almost triangular matrix. Also, the structure of the quantities to be found from this system will be of the form

$$Y_0 = F_0 \psi^{-1}(\alpha), \qquad Y_l = Y_0 X_l + F_l \qquad (l = 1, 2, \ldots, N) \tag{3.4}$$

For example, $\psi(\alpha)$, F_l and X_l for $N = 1$ are respectively

$$\psi(\alpha) = \frac{\pi}{2} - \alpha\varkappa_1\left(C_0 - \frac{C_1}{3}\alpha^2 + \frac{2\varkappa_1}{45\pi}C_1^2\alpha^5\right) \tag{3.5}$$

$$F_0 = f_0 + \frac{4}{3\pi}\varkappa_1 C_1 f_1 \alpha^3, \qquad X_1 = \frac{4\varkappa_1 C_1}{3\pi}\alpha^3, \qquad F_1 = \frac{40}{\pi} f_1$$

For purposes of brevity the solution of (3.1) for cases when $N > 1$ is not given.

Having found the coefficients Y_m ($m = 0, 1, \ldots$) and substituted them in (1.18), we obtain from (1.10) and (1.16) a formula for the contact stress

$$p(r) = a^{-1}\chi(r/a) \tag{3.6}$$

We have thus found a solution for the general case in the form of an infinite series. However, if the function (1.9) is a polynomial, then the key series is lost. As an example, let us find the solution $\chi_\alpha(t)$ of Equation (1.17) for $f(x) \equiv 1$. In this case $f_0 = F_0 = 1$, $f_l = F_l = 0$, $l = 1, 2, \ldots$, and consequently, taking into account (1.18), (3.2), (3.4), we obtain

$$\chi_\alpha(t) = \psi^{-1}(\alpha)(1 - t^2)^{-1/2}\sum_{m=0}^{N} X_m P_m^*(t), \qquad X_0 = 1 \tag{3.7}$$

We shall now derive a formula for evaluating the force P pressing the elastic body onto the foundation. We first substitute (3.6) into the integrand in the formula

$$P = 2\pi\int_0^a r p(r)\,dr \tag{3.8}$$

After carrying out the integration, taking into account (1.18) and the orthogonality of the Legendre polynomials, we obtain

$$P = 2\pi a Y_0 \tag{3.9}$$

We can determine z_0 from Equation (3.9) if we know the dimensions of the area of contact. This will not be possible in the case of only partial contact [1,2]. In this case the contact stress must be finite at $r = a$, which from (3.7) and (1.18) will be so if

$$\sum_{m=0}^{\infty} Y_m P_{2m}(0) = 0 \qquad (3.10)$$

Equations (3.9) and (3.10) enable us to find z_0 and a.

4. The solution derived above is an infinite series; it would be interesting to try to find a solution of the integral equation (1.6) with the kernel (1.15) in closed form.

With this aim in mind we make use of the well-known result [12]: that in order to obtain a solution of an integral equation with an arbitrary right-hand side we can find a solution $q_\alpha(x)$ of this equation, but with a right-hand side $f(x) \equiv 1$, and then make use of the formulas

$$\varphi(x) = \gamma(\alpha)\, q_\alpha(x) - \int_x^\alpha q_u(x)\, \gamma'(u)\, du \qquad (4.1)$$

where

$$\gamma(\alpha) = \frac{1}{M'(\alpha)} \frac{d}{d\alpha} \int_0^\alpha \frac{sf(s)}{q_\alpha^{-1}(s)}\, ds, \qquad M(\alpha) = \int_0^\alpha x q_\alpha(x)\, dx \qquad (4.2)$$

Here and in our following work a stroke will be used to denote a derivative.

It is not difficult to show that in the present case

$$q_\alpha(x) = \alpha^{-1} \chi_\alpha(x / \alpha) \qquad (4.3)$$

and consequently, Formulas (1.10), (4.1), (4.4) and (3.17) together give a solution to the problem in closed form. If we now start from the solution in closed form, Formula (3.8) becomes

$$(\theta_0 + \theta_1) P = 2\pi h M(\alpha)\, [z_0 - J(\alpha)] \qquad (4.4)$$

where

$$J(\alpha) = \int_0^1 (1 - t^2)^{-1/2} [z_1(at) + z_2(at)] \sum_{m=0}^{N} X_m P_m^*(t)\, dt \quad (a = \alpha h) \qquad (4.5)$$

If we substitute (4.3) in the second formula of (4.2), taking into account (1.18), we find that

$$M(\alpha) = \alpha \psi^{-1}(\alpha) \qquad (4.6)$$

Instead of (3.10), the condition of finiteness of the contact stress now becomes $\gamma(\alpha) = 0$, which, by substituting (4.3) in the first formula of (4.2) and taking into account (1.18) and (1.9), we can reduce to the form

$$z_0 = \frac{1}{M'(\alpha)} \frac{d}{d\alpha} M(\alpha) J(\alpha) \tag{4.7}$$

We obtain the equation for finding the dimensions of the area of contact by substituting (4.7) into (4.4).

We shall illustrate the application of the formulas obtained so far by two particular cases.

Suppose that an elastic foundation is indented by a stamp with a flat base. Then $\theta_0 = 0$, $z_0(r) \equiv z_1(r) \equiv 0$, and substitution of $f(x) = z_0\theta_1^{-1}$ into the first formula of (4.2) gives $\gamma(a) = z_0\theta_1^{-1}$. Consequently, taking into account (4.1) and (1.10), we obtain

$$p(r) = z_0\theta_1^{-1}a^{-1}\chi_\alpha(r/a) \tag{4.8}$$

If we confine ourselves to the first approximation (3.5), then

$$2\pi p(r) = P(a^2 - r^2)^{-1/2}\left[1 + \pi^{-1}C_1ah^{-3}\left(\frac{4}{3}a^2 - 2r^2\right)\right] \tag{4.9}$$

The settlement of the stamp is given by the formula

$$2\pi a z_0 = P\theta_1\psi(\alpha), \quad \alpha = a/h \tag{4.10}$$

which we obtain from (4.4), taking into account that $J(\alpha) = 0$.

As a second example, confining ourselves again to a first approximation, let us consider the indentation of an elastic body in the case when (compare [1])

$$z_0(r) + z_1(r) = Cr^2 \tag{4.11}$$

In this case we obtain the following formula for the contact stress

$$\pi p(r) = 1.5\,Pa^{-3}\sqrt{a^2 - r^2} \tag{4.12}$$

by making use of Expressions (3.6), (3.10) and (3.9) and bearing in mind that $N = 1$.

From an equation obtained by substituting (4.7) into (4.4) and taking into account (4.11), (4.5) and (3.5) we find that

$$a^3 = 1.5\pi P(\theta_0 + \theta_1)(8\pi C + P\theta_1 C_1 h^{-3})^{-1} \tag{4.13}$$

Similarly, from Equation (4.4) we obtain

$$Z_0 = (2\pi a)^{-1} P\,(\theta_0 + \theta_1)\,\psi(\alpha) + \frac{2}{3} C a^2 \left(1 - \frac{4}{15}\varkappa_1 C_1 \alpha^3\right) \qquad (4.14)$$

Here $\psi(\alpha)$ can be determined from the first of formulas (3.5).

In order to establish the number N of terms in the series (1.13) that should be retained (i.e. the degree of approximation required) in order to obtain a sufficiently accurate solution in any given case, we must determine the values of the function $\chi_\alpha(t)$, in terms of which, as is clear from our preceding work, the general solution to the problem is expressed.

TABLE 1.

	$\chi_\alpha(t)$						$\psi(\alpha)$		α
N	$t=0$	0.2	0.4	0.6	0.8	1	above theory	ref. [4]	
1	1.02	1.02	1.00	0.984	0.954	0.917	1.05		
2	1.01	1.01	1.00	0.983	0.959	0.931	1.04	1.04	0.5
3	1.01	1.01	1.00	0.983	0.959	0.929	1.04		
1	1.39	1.38	1.33	1.24	1.13	0.978	0.895		
2	1.36	1.34	1.30	1.24	1.18	1.11	0.846	—	0.75
3	1.37	1.35	1.31	1.25	1.17	1.07	0.857		
1	1.86	1.82	1.69	1.49	1.20	0.831	0.851		
2	1.77	1.74	1.65	1.56	1.48	1.51	0.654	0.715	1.0
3	1.81	1.78	1.71	1.58	1.37	1.12	0.730		

Some results applicable to the case (1.3), and approximations (N = 1, 2, 3) with the values (a) given in the footnote on page 210 and with $\theta_0 = 0$, are listed in Table 1, which gives the values not of the actual function $\chi_\alpha(t)$, which becomes infinite when $t = 1$, but the values of the function $\chi_\alpha(t) = \sqrt{(1 - t^2)}\,\chi_\alpha(t)$.

This table also gives values of the function $\psi(\alpha)$, for the same case and for the three approximations, which according to (4.10) determines the settlement of the stamp to an accuracy of a multiplier. For comparison, the table also gives values of this function derived from numerical results cited in [4].

On the basis of the results listed in Table 1, we conclude that in the case of a foundation in the form of an elastic layer, and with $\alpha \leqslant 0.5$, the solution of the contact problem need not be taken beyond a first approximation.

5. Let us now consider the case when there is no axial symmetry, when (1.2) is replaced by a more general relation

$$G(x) = x^{\nu}[1 + o(1)] \qquad \text{as} \quad x \to \infty \left(-1 < \nu < \tfrac{1}{2}\right) \tag{5.1}$$

Suppose that an elastic foundation of the type (1.1) is compressed by a stamp, the surface of the base of which is given by the equation $z = g(r, \phi)$. We shall assume that the function g can be expressed in the form

$$g(r, \varphi) = g_0(r) + \sum_{n=1}^{\infty} [g_n(r) \cos n\varphi + g_n^{-}(r) \sin n\varphi] \tag{5.2}$$

The contact stress $p(r, \phi)$ in this case can then be determined simply by finding the stress for the case*

$$g(r, \varphi) = g_n(r) \cos n\varphi \tag{5.3}$$

The contact stress will be of an analogous form

$$p(r, \varphi) = p_n(r) \cos n\varphi \tag{5.4}$$

In order to find the integral equation for the sought function P_n, we require a formula which will enable us to find the settlement $w^n(r, \rho)$ of points on the surface of the foundation under a load concentrated along the circumference of a circle of radius ρ and distributed according to a cosine law. Following the method given in [7], we can show that

$$w^{(n)}(r, \rho) = \theta_1 \rho \int_0^{\infty} G(th) J_n(rt) J_n(\rho t)\, dt \cos n\varphi \tag{5.5}$$

Taking into account (5.3) to (5.5) we find that

$$\int_0^{\alpha} K_n(x, \xi)\, \xi p_n^*(\xi)\, d\xi = g_n^*(\xi) \qquad (\xi \leqslant \alpha) \tag{5.6}$$

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* For the case when $g(r, \phi) = g_n - (r) \sin n\phi$ the same procedure is adopted, the resulting formulas differing only in a multiplier ($\cos n\phi$ becomes $\sin n\phi$). In passing, we note that the idea used here of reducing a three-dimensional contact problem to one possessing cyclic symmetry is taken from the thesis by V.I. Mossakovskii (Moscow, Institute of Mechanics, Soviet Academy of Sciences, 1953).

Here

$$p_n^*(\xi) = hp_n(h\xi), \qquad g_n^*(\xi) = \theta_1^{-1} g_n(h\xi) \tag{5.7}$$

$$K_n(x, \xi) = \int_0^\infty G(s) J_n(xs) J(\xi s)\, ds \tag{5.8}$$

We give below an approximate method of solving Equation (5.6) based on the following approximation of its kernel

$$K_n(x, \xi) \approx k^{(n)}(x, \xi) - (x\xi)^n H_n(x, \xi) = K^*(x, \xi) \tag{5.9}$$

where

$$k^{(n)}(x, \xi) = \int_0^\infty s^\nu J_n(xs) J_n(\xi s)\, ds \tag{5.10}$$

$$H_n(x, \xi) = \sum_{k=0}^{N} \frac{(-1)^k C_{k+n}^{(\nu)}}{4^{k+n} k!\,(k+n)!} M_k^{(n)}(x, \xi) \qquad (N = 1, 2, \ldots) \tag{5.11}$$

$$C_m^{(\nu)} = \int_0^A [s^\nu - G(s)]\, s^{2m}\, ds, \qquad C_m^{(0)} = C_m \qquad (m = 0, 1, 2, \ldots) \tag{5.12}$$

Approximation (5.9) is derived from the same considerations as (1.15). The numbers N and A have their former meaning, and the function $M_k^{(n)}(x,\xi)$ is determined from Fromula (1.14).

In order to find a solution to Equation (5.6) with the kernel $K_n^*(x,\xi)$, we first solve the integral equation (2.5) with kernel (5.10). To do so, bearing in mind the result used above, taken from [12], it is sufficient to solve the equation

$$\int_0^\alpha k^{(n)}(x, \xi)\, \xi q_\alpha(\xi)\, d\xi = 1 \qquad (\xi \leqslant \alpha) \tag{5.13}$$

and then apply Formulas (4.1) and (4.2).

The solution of Equation (5.13) can be found in the following way. We write the kernel of Equation (5.13), given by Formula (5.10), as follows

$$k^{(n)}(x, \xi) = \xi^{-1-\nu} k_\nu(x/\xi), \qquad k_\nu(y) = \int_0^\infty J_n(s) J_n(sy)\, s^\nu ds \tag{5.14}$$

We then make the substitutions

$$x = \alpha e^{-t}, \qquad \xi = \alpha e^{-\tau}, \qquad \alpha^{1-\nu} e^{-t} q_\alpha(\alpha e^{-t}) = \chi_\zeta(t)$$

and multiply both sides of Equation (5.13) by $a^\nu e^{-\nu t}$. We can then transform the latter to the following Wiener-Hopf integral equation of the first kind:

$$\int_0^\infty l(t-\tau)\,\chi_\zeta(\tau)\,d\tau = e^{i\zeta x}, \qquad \operatorname{Im}\zeta > 0 \qquad (l(y) = e^{-\nu y}k_\nu(e^{-y})) \tag{5.15}$$

After constructing the function $\chi_\zeta(t)$ by a method described in detail in [7] and illustrated by an example of an equation analogous to (5.15), we find that

$$q_\alpha(x) = 2^{1-\nu}\Gamma\left(1+\frac{n}{2}\right)\Gamma^{-1}\left(\mu+\frac{n}{2}\right)\Gamma^{-1}(\mu)\,x^n\Big[\alpha^{-n}(\alpha^2-x^2)^{\mu-1} + {} + n\int_x^\alpha \eta^{-1-n}(\eta^2-x^2)^{\mu-1}d\eta\Big] \qquad \left(\mu=\frac{1+\nu}{2}\right) \tag{5.16}$$

After substituting (5.16) in Formulas (4.1) and (4.2) and carrying out the necessary computations, we obtain a solution to Equation (1.6) with kernel (5.10) in the following form

$$\varphi(x) = \frac{2^{1-\nu}x^n}{\Gamma^2(\mu)}\left[\frac{\Phi(\alpha)}{(\alpha^2-x^2)^{1-\mu}} - \int_x^\alpha \frac{\Phi'(u)\,du}{(u^2-x^2)^{1-\mu}}\right] \qquad \left(\mu=\frac{1+\nu}{2}\right) \tag{5.17}$$

Here,

$$\Phi(\alpha) = \alpha^{-2n-\nu}\frac{d}{d\alpha}\alpha^{n+2\mu}\int_0^1 \frac{s^{n+1}f(s\alpha)}{(1-s^2)^{1-\mu}}\,ds \tag{5.18}$$

Formulas (5.17) and (5.18) can be written in the symbolic form

$$\varphi = Lf \tag{5.19}$$

Here L is an operator which transforms the function f into a solution of the integral equation.

6. In order to solve the fundamental equations (5.6), we make use of Carleman's method, which in this particular case can be described as follows. Substitute in (5.6), not the kernel, but its expression (5.9) in the form of the sum of two functions, and then take the second of them over to the right-hand side. As a result we obtain an equation which has a solution in the form of Formula (5.19), but with a right-hand side given by

$$f(x) = g_n^*(x) + \int_0^\alpha (x\xi)^n H_n(x,\xi)\,p_n^*(\xi)\,d\xi \tag{6.1}$$

If we now expand (5.11) into a Maclaurin series in ξ, we find that

$$H_n(x,\xi) = \sum_{m=0}^{N} h_m^{(n)}(x)\,\xi^{2m} \tag{6.2}$$

Here

$$h_m^{(n)}(x) = \sum_{k=m}^{N} \frac{(-1)^k 4^{-k-n} C_{k+n}^{(\nu)} x^{2(k-m)}}{(n+m)!\,(n+k-m)!\,(k-m)!\,m!}$$

We introduce the further notation

$$Y_m^{(n)} = \int_0^a \xi^{2m+n+1} p_n^*(\xi)\,d\xi \qquad (m = 0, 1, \ldots, N) \tag{6.3}$$

Taking into account (6.2) and (6.3), we can replace (6.1) by

$$f(x) = x^n \sum_{m=0}^{N} h_m^{(n)}(x)\,Y_m^{(n)} + g_n^*(x) \tag{6.4}$$

If we now apply the operator L to the function (6.4), we obtain*

$$p_n^*(x) = Lf \tag{6.5}$$

The numbers $Y_m^{(n)}$, which according to (6.4) occur on the right-hand side of (6.5), can be found, for example, in the following way. Multiply both sides of Expression (6.5) by x^{2l+n+1}, and, taking into account (6.3) and Formulas (5.17), (5.18) which determine the operator L, integrate with respect to x over the interval $(0, a)$. As a result we obtain the following set of equations for determining $Y_l^{(n)}$:

$$Y_l^{(n)} = \frac{2^{-\nu}(n+l)!}{\Gamma^2(\mu)\,(\mu)_{n+l}} \left[\frac{a^{2(l+n+\mu)}}{4^n (l+n+\mu)} \sum_{m=0}^{N} A_{ml}^{(n)} Y_m^{(n)} + g_l^{(n)} \right]$$

$$(l = 0, 1, \ldots, N) \tag{6.6}$$

Here

$$A_{ml}^{(n)} = \frac{l+n+\mu}{(n+m)!\,m!} \sum_{k=m}^{N} \frac{(-1)^k 4^{-k} C_{k+n}^{(\nu)} a^{2(k-m)}}{(l+n+k-m+\mu)\,(k-m)!\,(\mu)_{n+k+m}} \tag{6.7}$$

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* If we take into account definition (6.3), it is not difficult to show that Expression (6.5) is a Fredholm equation of the second kind with a degenerate kernel. When the values of $Y_m^{(n)}$ have been found by some method or other, the above relation constitutes a formula for finding the functions p_n^* ($n = 0, 1, 2, \ldots$), in terms of which the required contact stress can be expressed.

$$g_l^{(n)} = 2\int_0^\alpha \Phi_n^*(u)\, u^{2(l+n)+\nu}\, du, \qquad (z)_n = z(z+1)(z+2),\ldots,(z+n-1)$$
$$(z)_0 = 1$$

Here Φ_n^* denotes a function obtained from Formula (5.18) by replacing f by g_n^*.

Let us consider the more detailed case when an elastic foundation is indented by a stamp with a flat base under the action of an eccentrically applied force P (the eccentricity being denoted by e).

In this case Expansion (5.2) consists of two terms, where $g_0(r) = z_0$ $g_1(r) = \beta r$, and z_0 has its former meaning, and β is the angle of inclination of the stamp after deformation (compare [13,6]). The contact stress now has the form

$$p(r, \varphi) = p_0(r) + p_1(r)\cos\varphi \tag{6.8}$$

In the present case, according to (5.7)

$$g_n^*(x) = \gamma_n x^n \ (n = 0,1), \qquad \gamma_0 = z_0\theta_1^{-1}, \qquad \gamma_1 = \beta h\theta_1^{-1} \tag{6.9}$$

Taking this into account, and introducing the new unknowns

$$X_l^{(n)} = 4^{-n}(\mu)_n \gamma_n^{-1} Y_l^{(n)} \qquad (n = 0, 1) \tag{6.10}$$

(6.4) is replaced by

$$f(x) = \gamma_n x^n \Big[1 + 4^n (\mu)_n^{-1} \sum_{m=0}^{N} h_m^{(n)}(x)\, X_m^{(n)} \Big] \tag{6.11}$$

Substituting this expression in (5.18), we obtain

$$\Phi(\alpha) = \gamma_n (\mu)_n^{-1} \Lambda_n(\alpha), \ \Lambda_n(\alpha) = 1 + \sum_{m=0}^{N} X_m^{(n)} E_m^{(n)} \qquad (n = 0, 1) \tag{6.12}$$

Here

$$E_m^{(n)} = \frac{(-1)^m 4^{-m}}{(n+m)!\, m!} \sum_{j=0}^{N-m} \frac{(-1)^j C_{n+m+j}^{(\nu)}}{4^j j!\, (\mu)_{n+j}} \alpha^{2j} \tag{6.13}$$

We introduce the polynomial

$$p_m^{(\mu)}(x) = \frac{1}{2} \frac{m!}{(\mu)_{m+1}} \sum_{k=0}^{m} \frac{(\mu)_{m-k}}{(m+k)!} x^{2k} \tag{6.14}$$

We can then write

$$\int_x^\alpha \frac{u^{2j-1}\,du}{(u^2-x^2)^{1-\mu}} = \alpha^{2(j-1)}(\alpha^2-x^2)^\mu\, p_{j-1}^{(\mu)}(x/\alpha) \qquad \left(\mu = \frac{1+\nu}{2}\right) \tag{6.15}$$

Applying the operator L to the function (6.11) and taking into account Formula (5.17), we find that

$$p_n^*(x) = \frac{2^{1-\nu}\gamma_n x^n}{\Gamma^2(\mu)(\mu)_n}(\alpha^2-x^2)^{\mu-1}\Big\{1+\sum_{m=0}^{N} X_m^{(n)}\Big[E_m^{(n)} - $$

$$-\frac{(-1)^m 2^{1-2m}}{(n+m)!\,m!}\sum_{j=1}^{N-m}\frac{(-1)^j C_{n+m+j}^{(\nu)}\cdot\alpha^{2(j-1)}}{4^j(j-1)!\,(\mu)_{n+j}}\,\frac{p_{j-1}^{(\mu)}(x/\alpha)}{(\alpha^2-x^2)^{-1}}\Big]\Big\} \qquad (n=0,1) \tag{6.16}$$

The set of equations

$$X_l^{(n)} = \frac{2^{-2n-\nu}(l+n)!}{\Gamma^2(\mu)(\mu)_{n+l+1}}\Big[1+\sum_{m=0}^{N} X_m^{(n)} A_{ml}^{(n)}\Big]\alpha^{2(l+n+\mu)} \qquad (l=0,1,\ldots,N) \tag{6.17}$$

for determining the numbers $X_l^{(n)}$ can be obtained from (6.6) on the basis of (6.10) and (6.9).

If we are given the eccentricity e and the magnitude of the force P, then z_0 and β can be found from the conditions of equilibrium of the stamp (compare [13])

$$Pe^n = \int_0^a\int_0^{2\pi} p(r,\varphi)\cos^n\varphi\, r^{n+1}\,d\varphi\,dr = 2^{1-n}\pi h^{1+n}\int_0^\alpha p_n^*(x)\,x^{1+n}\,dx \qquad (n=0,\ 1) \tag{6.18}$$

In order to check the validity of the second equality we make use of (6.8) and (5.7). From Relations (6.18), and taking into account (6.3), (6.10) and (6.9), we find that

$$z_0 = \frac{P\theta_1}{2\pi h}\,\frac{1}{X_0^{(0)}}, \qquad \beta = \frac{P\theta_1\varepsilon\cdot\mu}{4\pi h^2 X_0^{(1)}} \qquad \left(\varepsilon = \frac{e}{h}\right) \tag{6.19}$$

It is of interest to derive a formula for the maximum eccentricity e_{max} possible without separation of the stamp from the foundation. In this case, following Abramov [13], we obtain

$$\varepsilon_{\max} = h^{-1}e_{\max} = 2\Lambda_0(\alpha)\,X_0^{(1)}\,[\Lambda_1(\alpha)\ X_0^{(0)}]^{-1} \tag{6.20}$$

It will be noted from the above calculations that it is simpler to solve the set of equations (6.17) by the method of successive approximations. Convergence is particularly rapid for $n = 1$ (three or four approximations are sufficient to obtain an answer to the accuracy of four places). If $n = 0$ the convergence is less rapid.

However, there is no need to solve (6.17) when $n = 0$ and $\nu = 0$, since the numbers $X_m^{(n)}$ are related in a sufficiently straightforward manner with the solution of a simpler set of equations, namely (3.1) with $f_l = 1$, $l = 0$, $f_l = 0$, $l > 0$, i.e. with the numbers X_m occurring in Expression (3.7) for the function $\chi_\alpha(t)$, which is related to $p_0{}^*(x)$ by the expression

$$p_0{}^*(x) = \gamma_0\alpha^{-1}\chi_\alpha(x/\alpha) \qquad (6.21)$$

After substituting (6.21) into (6.3) and taking into account (3.7), (3.3) and (6.10), we obtain

$$X_m^{(0)} = \alpha^{1+2m}\psi^{-1}(\alpha)\sum_{k=0}^{N} X_k b_m^{(k)} \qquad (6.22)$$

The values of $b_m^{(k)}$ are given by Formula (3.3).

TABLE 2.

α	0.5			0.75			1.0		
N	1	2	3	1	2	3	1	2	3
$10\cdot X_0^{(1)}$	0.137	0.137	0.137	0.483	0.490	0.489	1.16	1.29	1.24
ε_{max}	0.161	0.162	0.162	0.223	0.237	0.233	0.254	0.330	0.288

Table 2 shows some results obtained by using the above formulas. It gives values for three approximations $N = 1, 2, 3$, of the quantity $X_0^{(1)}$, the inverse of which, according to (6.19), determines the angle of inclination of the stamp to the accuracy of a multiplier. This table also gives values of ϵ_{max} calculated from Formula (6.20). The calculations were carried out for a foundation in the form of an elastic layer ($\nu = 0$) using the values of C_m given by (a) in the footnote on page 210. It will be seen from Table 2 that in this case, as well as in the axisymmetric problem, if $\alpha < 0.5$ we need not go beyond a first approximation.

In conclusion, we should point out that the method described above can be applied to the contact problem for a thin circular plate on a foundation of any sort.

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